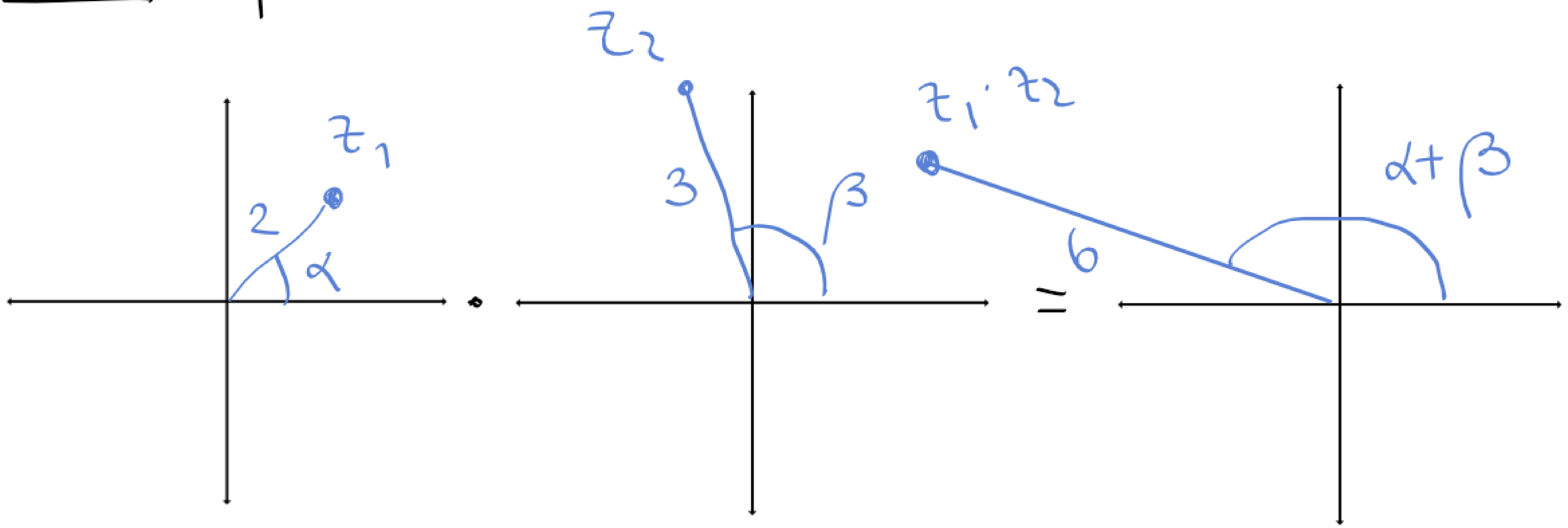
$\mathbb{C}$: opáčko – násobení



$$z_1 = 2\,(\cos\alpha + i\sin\alpha)$$

$$z_2 = 3\,(\cos\beta + i\sin\beta)$$

$$z_1 \cdot z_2 = 6\,\big(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\big).$$

Posloupnosti $\mathbb{C}$-čísel

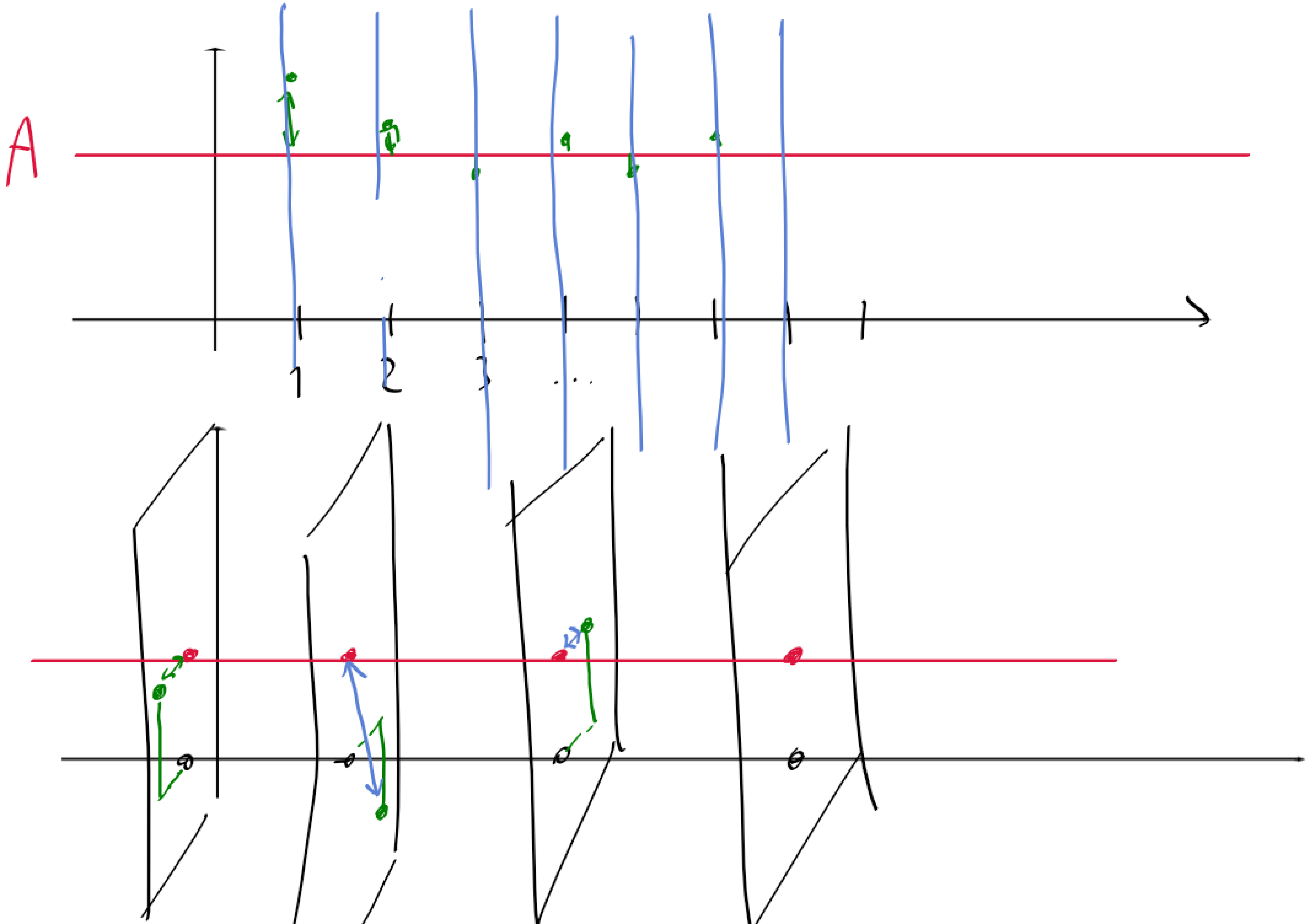
Formálně jde o zobrazení (funkci) z $\mathbb{N}$ do nějaké množiny – zde $\mathbb{C}$.

Speciální $f: \mathbb{N} \longrightarrow \mathbb{C}$. (def. obor je $\mathbb{N}$)

Značíme: $z_n := f(n)$, resp. $\{z_n\}_{n=1}^{\infty} = f$.

Definice: · Připomenutí $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$:

$$\lim_{n\to\infty} x_n = A \in \mathbb{R} \overset{def.}{\iff} \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 : |x_n - A| < \varepsilon.$$



Jde o to, že vzdálenost x_n od $A \xrightarrow{n\to\infty} 0$.

Buďiž $\{z_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$: $\lim z_n = z \in \mathbb{C}$

$\overset{\text{def.}}{\iff} \forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 : \underbrace{|z_n - z|}_{\text{vzdálenost } z_n, z} < \varepsilon$.

$$\left(|z| = |a+bi| = \sqrt{a^2+b^2} \right).$$

<u>Lemma</u>: Je-li $\{z_n\}_{n=1}^{\infty}$ posl. $\mathbb{C}$-čísel, $z \in \mathbb{C}$, pak NVJE:

(i) $\lim\limits_{n\to\infty} z_n = z \in \mathbb{C}$

(ii) $\lim \operatorname{Re}(z_n) = \operatorname{Re}(z)$ a zároveň
$\lim \operatorname{Im}(z_n) = \operatorname{Im}(z)$.

Tj. při interpretaci jako $\mathbb{R}^2 \simeq \mathbb{C}$
$(a_n, b_n) \to (a, b) \iff a_n \to a \wedge b_n \to b$.

<u>Důkaz</u>: (i) $\Rightarrow$ (ii) : Všimněme si, že

$$|\operatorname{Re}(a+bi)| \leq |a+bi| \qquad (a, b \in \mathbb{R})$$
$$|\operatorname{Im}(a+bi)| \leq |a+bi|.$$

Nechť platí (i) tj. $\overset{\text{def.}}{\iff}$ ($\lim z_n = z$) $|z_n - z| \to 0$.

Potom $\underbrace{0}_{\to 0} \leq \underbrace{|\overbrace{\operatorname{Re}(z_n - z)}^{\operatorname{Re}(z_n) - \operatorname{Re}(z)}|}_{\to 0} \leq \underbrace{|z_n - z|}_{\to 0} \to 0$

Podle L. o 2 policajtech $\to 0$, $n \to \infty$.

Podobně $|\operatorname{Im}(z_n - z)| \to 0$.

$\overset{1.\text{SEM}}{\Longrightarrow} \lim \operatorname{Re}(z_n) = \operatorname{Re}(z)$
$\lim \operatorname{Im}(z_n) = \operatorname{Im}(z)$.

Víme $\operatorname{Re}(z+w) = \operatorname{Re}(z) + \operatorname{Re}(w)$, podobně Im.

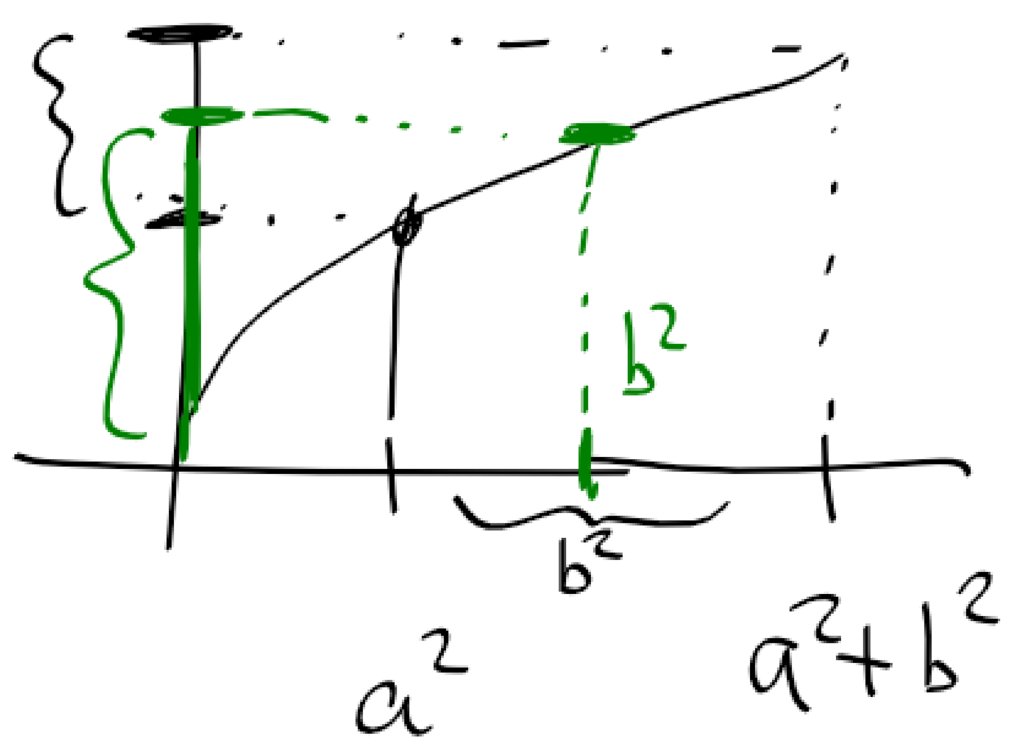
(ii) $\Rightarrow$ (i): $0 \leq |z_n - z| \leq \underbrace{|\operatorname{Re}(z_n - z)|}_{\to 0} + \underbrace{|\operatorname{Im}(z_n - z)|}_{\to 0}$ podle (ii).

Podle LO2P $\underbrace{|z_n - z|}_{} \to 0$, $n \to \infty$.

Zbývá dokázat $|z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$

Nechť $z = a + bi$, $a, b \in \mathbb{R}$.

$$\underline{|z|} = \sqrt{a^2+b^2} \overset{\text{w.}}{\leq} \sqrt{a^2} + \sqrt{b^2} = |a| + |b| \qquad \boxtimes$$



konkavita.

Chceme

Důsledek: lim Re z_n = Re lim z_n
lim Im z_n = Im lim z_n

umíme – posl. $\mathbb{R}$-čísel.

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[755] • $$\lim_{n\to\infty} \frac{4n-i}{\underbrace{2}_{a}+\underbrace{in}_{b}} = \lim_{n\to\infty} \frac{4n-\sqrt{-1}}{2+n\cdot\sqrt{-1}} =$$

$$= \lim_{n\to\infty} \frac{(4n-i)(2-in)}{\underbrace{(2+in)}_{z}\underbrace{(2-in)}_{\overline{z}}} = \lim_{n\to\infty} \frac{8n - n - 2i - 4in^2}{4-(in)^2} =$$

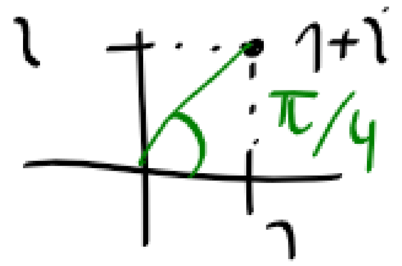
$z \cdot \overline{z} = |z|^2$

$$= \lim_{n\to\infty} \frac{7n - i\cdot(2+4n^2)}{4+n^2} = \quad \text{Lemma.}$$

$$= \lim \frac{7n}{4+n^2} - i \lim \frac{4n^2+2}{n^2+4} =$$

$$= 0 - i\cdot 4 = \underline{\underline{-4i}}$$

- $$\lim_{n\to\infty} \frac{i}{n} \cdot \Big(\underbrace{\frac{1+i}{\sqrt{2}}}_{z} \Big)^n = (*)$$



Řešení a):

$$|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$$

$$z = |z| \cdot \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$z^n = \cos\left(n\cdot\frac{\pi}{4}\right) + i\sin\left(n\cdot\frac{\pi}{4}\right)$$

$$(*) = \lim_{n\to\infty} \frac{i}{n} \cdot \left(\cos\left(n\frac{\pi}{4}\right) + i\sin\left(n\cdot\frac{\pi}{4}\right)\right) =$$

$$= \lim_{n\to\infty} \Bigg(\underbrace{\frac{-\sin\left(n\frac{\pi}{4}\right)}{n}}_{Re(\cdots)} + i \cdot \underbrace{\frac{\cos\left(n\frac{\pi}{4}\right)}{n}}_{Im(\cdots)} \Bigg) \overset{\text{Lemma}}{=}$$

$$= \lim\left(\frac{\cdots}{\cdots}\right) + i\lim\left(\cdots\right) =$$

$$= 0 + i\cdot 0 = 0.$$

„omezená · nulová = nulová"

b):

$$0 \leq \left|\frac{i}{n}\left(\frac{1+i}{\sqrt{2}}\right)^n\right| = \frac{1}{n}\cdot\left|\frac{1+i}{\sqrt{2}}\right|^n =$$

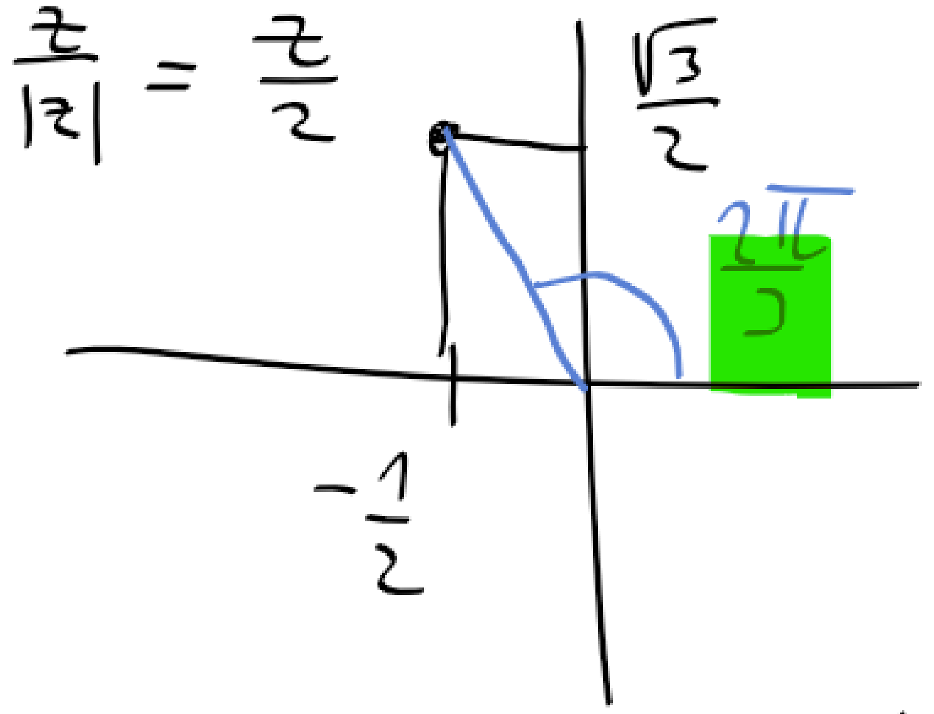
$$= \frac{1}{n}\cdot 1^n = \frac{1}{n} \xrightarrow{n\to\infty} 0.$$

Tedy $\lim \left|\frac{i}{n}\left(\frac{1+i}{\sqrt{2}}\right)^n - 0\right| = 0$

$$\overset{\text{def.}}{\iff} \lim \frac{i}{n}\left(\frac{1+i}{\sqrt{2}}\right)^n = 0.$$

• $\lim\limits_{n\to\infty} \dfrac{(-1+i\sqrt{3})^{3n+1}}{8^n-1} = (*)$ $\quad z := -1+i\sqrt{3}$

$\dfrac{z}{|z|} = \dfrac{z}{2}$



$$|z| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$z = 2\cdot\left(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right)$$

$$z^{3n+1} = 2^{3n+1}\cdot\left(\cos\left((3n+1)\frac{2\pi}{3}\right) + i\sin\left((3n+1)\frac{2\pi}{3}\right)\right) =$$

$$= 2\cdot 8^n\cdot\left(\cos\left(2n\pi+\frac{2\pi}{3}\right) + i\sin\left(2n\pi+\frac{2\pi}{3}\right)\right) =$$

$$= 2\cdot 8^n\cdot\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$(*) = \lim_{n\to\infty}\frac{1}{8^n-1}\left(2\cdot 8^n\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right) =$$

$$= \lim_{n\to\infty}\frac{\cancel{8^n}\cdot z}{\cancel{8^n}(1-8^{-n})} = \lim\frac{z}{1-8^{-n}} = (+)$$

$$= \lim\frac{-1+i\sqrt{3}}{1-8^{-n}} = \lim\frac{-1}{1-8^{-n}} + i\lim\frac{\sqrt{3}}{1-8^{-n}}$$

$$= \frac{-1}{1} + i\cdot\frac{\sqrt{3}}{1} = -1+i\sqrt{3} = z.$$

$$(+) \overset{\text{VOAL-C}}{=} \frac{\lim z}{\lim(1-8^{-n})} = \frac{z}{1} = z.$$

Řady $\mathbb{R}$-čísel

Symbol $\sum_{n=1}^{\infty} a_n$ resp. $a_1 + a_2 + a_3 + \dots$,

kde $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ je daná posl.

Jde o to, že ta čísla chceme sčítat.

Součet: značíme také $\sum_{n=1}^{\infty} a_n =: s$

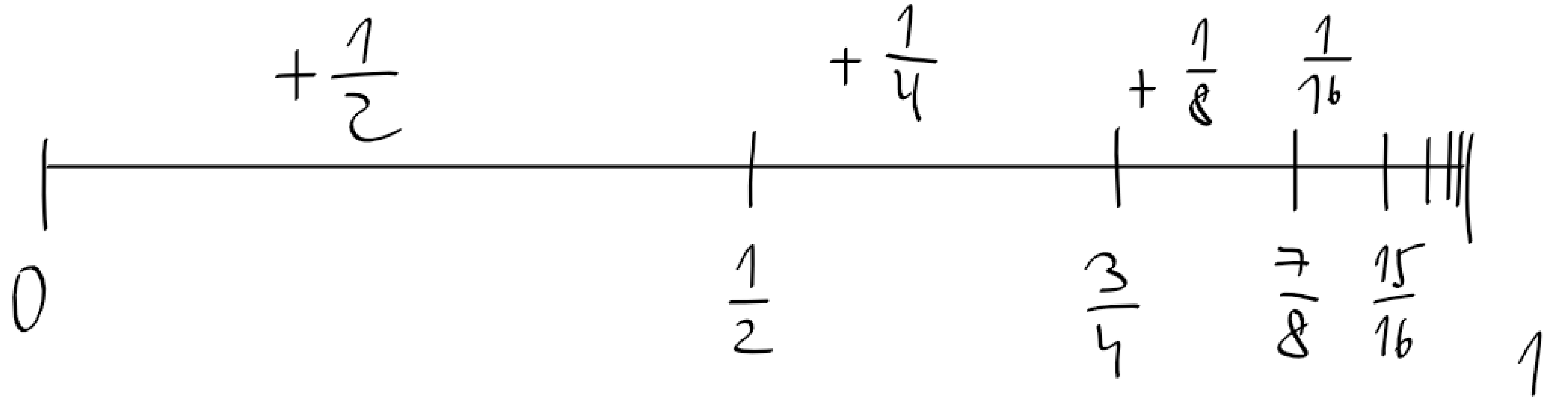
$$\overset{\text{def.}}{\iff} \lim_{N\to\infty} s_N = s, \quad \text{kde}$$

$$s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \dots + a_N$$

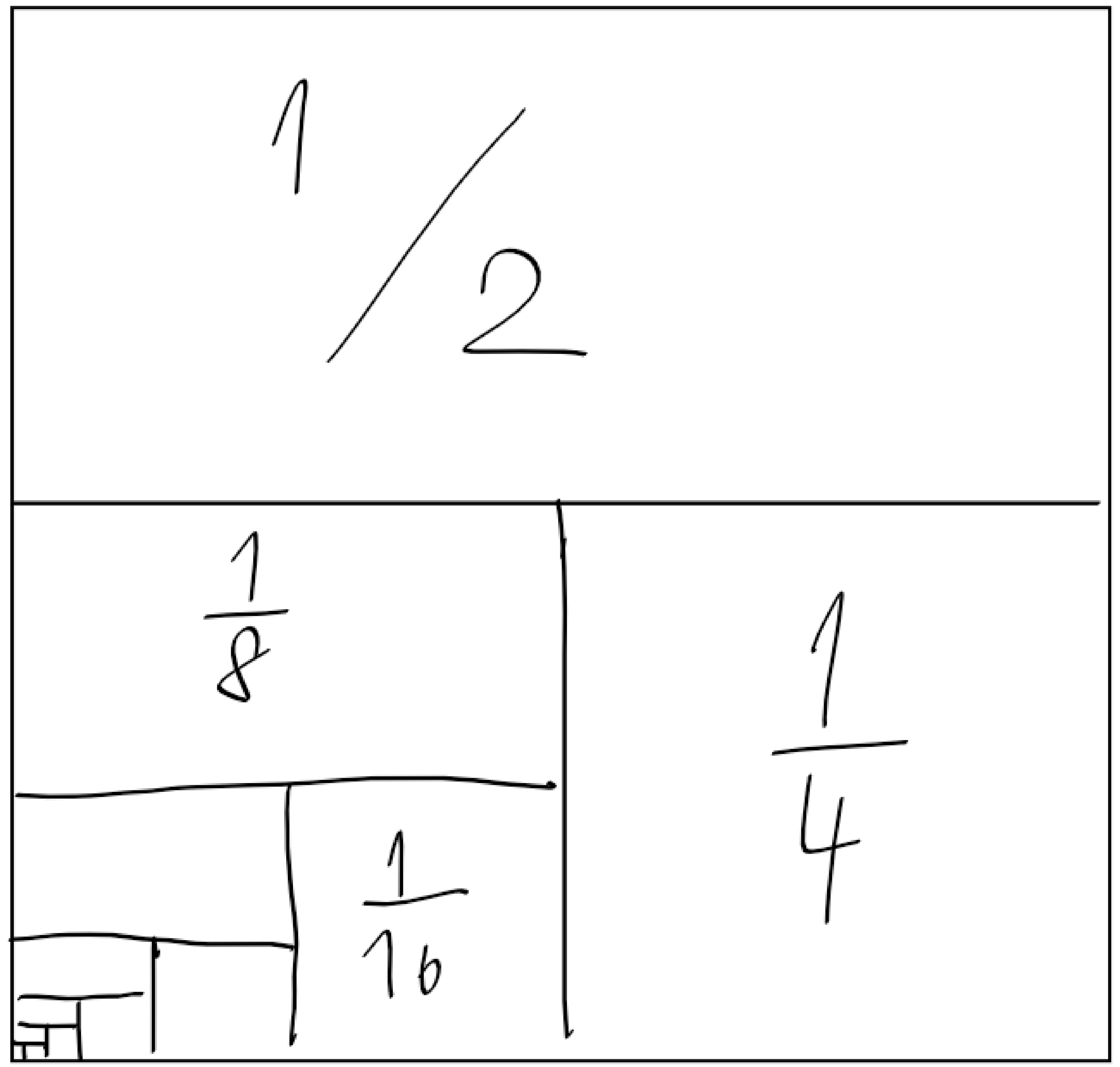
Pozorování: Pokud $\forall n \in \mathbb{N}: a_n \geq 0$,

pak $\sum_{n=1}^{\infty} a_n$ má součet $\in [0, \infty]$.

Příklad: $\sum_{n=1}^{\infty} \frac{1}{2^n}$



Snadné: $\sum_{n=1}^{N} \frac{1}{2^n} = 1 - \frac{1}{2^N}$ (indukcí)



Obecná geometrická řada: řada tvaru

$$\sum_{n=0}^{\infty} q^n \qquad (q \in \mathbb{R})$$

má konečný součet $\frac{1}{1-q}$ $\Longleftrightarrow$ $|q|<1$.

$$\left(\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \; |q|<1 \right)$$

<u>Důkaz</u>: nechť $q \in \mathbb{C}$, $|q|<1$.

$$s_N = \sum_{n=0}^{N} q^n = 1 + q + q^2 + \dots + q^N$$

<u>Platí</u>: $1-q^{N+1} = (1-q)(1+q+q^2+\dots+q^N)$

$$\Rightarrow \quad \frac{1-q^{N+1}}{1-q} = 1+q+q^2+\dots+q^N = s_N$$

$$\lim s_N = \lim_{N\to\infty} \frac{1-q^{N+1}}{1-q} =$$

$$= \frac{1}{1-q} \lim_{N\to\infty} \left(1-q^{N+1}\right) =$$

$$= \frac{1}{1-q}\left(1 - \lim q^{N+1}\right) = \frac{1}{1-q}(1-0)$$

$$\left[\lim_{N\to\infty} |q^{N+1}| = \lim \underbrace{|q|}_{\in [0,1) \subset \mathbb{R}}{}^{N+1} = 0 \right]$$

$\boxtimes$

$\longleftarrow$

<u>Příklad</u>: $0,\overline{9} = 1$

$$0,\overline{9} = 0,9 + 0,09 + \dots = 9\cdot(0,1+0,01+\dots) = 1$$

$$9 \cdot \sum_{n=1}^{\infty} 10^{-n} = 9 \cdot \sum_{n=0}^{\infty} \frac{1}{10}\cdot 10^{-n} = \frac{9}{10}\cdot \sum 10^{-n} = \frac{9}{10}\cdot\frac{1}{1-\frac{1}{10}}$$

$$\sum_{n=1}^{\infty} \sin n = \sin 1 + \sin 2 + \sin 3 + \dots$$

súčet neexistuje

$$\sum \frac{(-1)^n}{n} < \infty$$

$\sum (-1)^n$ súčet nemá (ω.)